Dispersive blow up for Schrödinger type equations

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Joint work with Jerry Bona (Chinese Annals of Maths. B, 2010)+ work in progress with J. Bona and C. Sparber

The aim is to study the formation and role of dispersive singularities in Schrödinger type equations and in linearized water waves equations (Euler with free boundary) which could be an explanation to the freak waves (rogue waves) formation.
More generally one would like to explain how most of dispersive equations (linear or not) may have very arbitrary large solutions at a prescribed point in space-time, initiating from an arbitrary small initial data.

Dispersive blow-up is a focussing type of behavior which is due to both the unbounded domain in which the problem is set and the propensity of the dispersion relation to propagate energy at different speeds. These two aspects allow the possibility that widely separated, small disturbances may come together locally in space-time, thereby forming a large deviation from the rest position. Possible relevance for explaining the genesis of rogue waves on the surface of large bodies of water and in electrical networks.

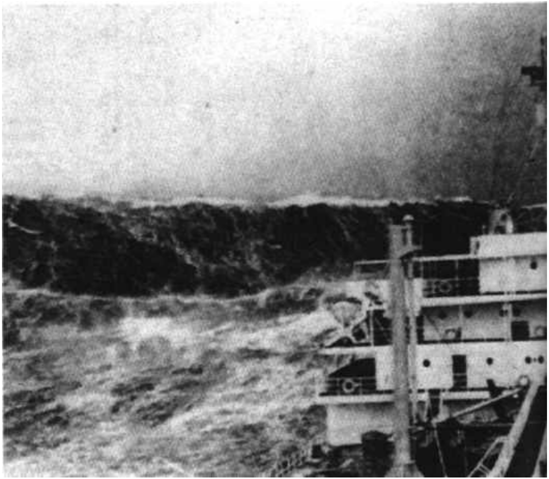
One of the proposed routes to rogue-wave formation is *concurrence*. This is the idea that the ambiant wave motion in a big body of water possesses a large amount of energy which could, in the right circumstances, temporarily coallesce in space, leading to giant waves.

Rogue waves, (freak waves), occur in both deep and shallow water. While the free surface Euler equations could be taken as the overall governing equations in both deep and shallow water, there is much to be learned from approximate models.
These, however, differ in deep and shallow water regimes. Our earlier work dealt with the shallow water situation, exemplified by the Korteweg-de Vries equation and Boussinesq-type systems of equations. Interest is focussed here on the deep water situation for surface water waves and on general equations of Schrödinger type.

Some references on freak waves.

- *Oceanic Rogue Waves* : Kristian Dysthe, Harald E. Krogstad, and Peter Müller, Annu. Rev. Fluid Mech. 2008 **40**, 287–310.
- Christian Kharif, Efim Pelinovsky et Alexey Slunyaev, *Rogue waves in the ocean*, Springer, Berlin, 2009

≪The term rogue or freak wave has long been used in the maritime community for waves that are much higher than expected for the sea state. Draper (1964, 1971) gave an early account of the phenomenon and introduced the terms to the scientific community, and Mallory (1974) provided the first discussion of the giant waves in the Agulhas current. He listed 12 reported hits and/or observations of abnormal waves (some of them causing severe damage) between 1952 and 1973. For the seafarer, rogue waves represent a frightening and often life-threatening phenomenon. There are many accounts of such waves hitting passenger ships (Didenkulova et al. 2006), container ships, oil tankers, fishing boats, and offshore and coastal structures, sometimes with catastrophic consequences. It is believed that more than 22 supercarriers have been lost because of rogue waves between 1969 and 1994 ≫.



The starting point is the classical paper of Benjamin, Bona and Mahony (1972) where they pointed out the "bad" behaviour of the "Airy" equation with respect to high frequencies (as Jerry pointed out, we should called it "Stokes equation" but that would add more confusion..).

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \\ u(.,0) = \phi \end{cases} \tag{1}$$

Take

$$\phi(x) = \frac{Ai(-x)}{(1+x^2)^m},$$

with

$$\frac{1}{8} < m < \frac{1}{4},$$

where Ai is the Airy function (now the name is correct though Airy did not introduced it in the context of water waves !).

Then $\phi \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$.
The solution $u \in C(\mathbb{R}_+; L^2(\mathbb{R}))$ is given by

$$\frac{c}{t^{\frac{1}{3}}} \int_{\mathbb{R}} Ai(\frac{x-y}{t^{\frac{1}{3}}}) \frac{Ai(-y)}{(1+y^2)^m} dy.$$

When $(x,t) \to (0,1)$, $u(x,t) \to c \int_{\mathbb{R}} \frac{Ai^2(-y)}{(1+y^2)^m} dy = +\infty$.

Actually one can prove with some extra work (this is not totally obvious) using the asymptotics of the Airy function (Bona-S. 1993) that
u is continuous on $\mathbb{R} \times \mathbb{R}_+^*$ except at $(x,t) = (0,1)$.

A similar "dispersive blow-up" holds true for the KdV equation (or any generalized KdV equation). For instance (Bona-S. 1993) :

Theorem

Let $(x*, t*) \in \mathbb{R} \times \mathbb{R}_+^*$*. There exists* $\phi \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *such that the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} + uu_x + \frac{\partial^3 u}{\partial x^3} = 0, \\ u(.,0) = \phi \end{cases} \tag{2}$$

has a unique solution $u \in C([0,\infty); L^2(\mathbb{R}) \cap L^2_{loc}(\mathbb{R}_+; H^1_{loc}(\mathbb{R}))$ *which is continuous on* $(\mathbb{R} \times \mathbb{R}_+^*) \setminus (x*, t*)$ *and satisfies*

$$\lim_{(x,t)\to(x*,t*)} |u(x,t)| = +\infty.$$

Sketch of proof : one can reduce to $(x*, t*) = (0, 1)$. We take ϕ leading to the dispersive blow-up of the linear Airy equation. Write the solution as

$$u(x,t) = S(t)\phi(x) + \int_0^t \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{1}{3}}} Ai\left(\frac{(x-y)}{(t-s)^{\frac{1}{3}}}\right) uu_x(y,s)dsdy$$

$$= S(t)\phi(x) + C\int_0^t \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{2}{3}}} Ai'\left(\frac{(x-y)}{(t-s)^{\frac{1}{3}}}\right) u^2(y,s)dsdy$$

This seems silly since Ai' grows as $(-x)^{\frac{1}{4}}$ as $x \to -\infty$.
The solution is to work in a weighted L^2 space and u^2 will compensate the growth of Ai'.

More precisely one can solves the Cauchy problem in the weighed space $L^2(\mathbb{R}, w)$ where

$$w(x) = w_\sigma(x) = \begin{cases} 1 & \text{for } x < 0 \\ (1+x^2)^\sigma & \text{for } x > 1. \end{cases}$$

Choosing $\frac{3}{16} < m < \frac{1}{4}$, the initial data $\phi \in L^2(\mathbb{R}, w_\sigma)$ where $\sigma \geq \frac{1}{16}$. The linear part still blows up at $(0,1)$. On the other hand the Duhamel integral is shown to be a continuous function of (x,t). So the nonlinear solution blows up exactly at $(0,1)$.

- Similar results for **any** generalized KdV equations and for the C^k norms (for more regular data).
- Alternative proof for modified KdV (Linares-Scialom 1993). Based on the existence proof via the KPV method and use that the Duhamel part is smoother than the free part. This smoothing effect is not true for KdV, but true for KP II (see Isaza-Mejia-Tzvetkov 2006).

The same method works for the linear Schrödinger equation, and more generally for linear dispersive equations with unbounded phase velocity , $\lim_{k\to\infty} c(k) = \frac{\omega(k)}{k} = \infty$.

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta u = 0, \\ u(.,0) = \phi \end{cases} \tag{3}$$

Of course it has been known for long time that the Schrödinger propagator is ill-posed in L^∞ (see *eg* Hörmander 1961, or the book *Partial differential equations* by Jeff Rauch for "abstract" results). The results below precise the structure of the dispersive blow-up and can be extended to the *nonlinear* case.

Theorem

Let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$ be given. There exist functions ϕ lying in the class $C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the corresponding solution u of (3) satisfies

1. $u \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^d))$,
2. *u is a continuous function of (x, t) on $\mathbb{R}^d \times ((0, +\infty) \setminus \{t^*\})$,*
3. *$u(\cdot, t^*)$ is a continuous function of x on $\mathbb{R}^d \setminus \{x^*\}$ and*
4. $$\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\ (x,t)\neq(x^*,t^*)}} |u(x,t)| = +\infty.$$

Remark

In particular, one deduces from the previous Theorem that for any fixed $t \in (0, +\infty) \setminus \{t^*\}$, the function $x \mapsto u(x, t)$ is continuous on $\mathbb{R}^d$ (but not necessarily bounded).

- In dimension 2 and higher, the initial data leading to DBU can belong to the Sobolev space $H^1(\mathbb{R}^n)$, or even be smoother.

Remark

With some modifications, a similar analysis applies to what is sometimes called the "hyperbolic" Schrödinger equation, namely

$$\begin{cases} i\partial_t u + \partial_{xx} u - \partial_{yy} u = 0, \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\ u(x,0) = \phi(x), \quad \text{for} \quad x \in \mathbb{R}^2. \end{cases} \tag{4}$$

This equation is the linearization about the rest state of a model for deep water surface gravity waves (Zakharov 1968).

- The dispersive blow-up is stable to smooth perturbations.
- Can be used (by truncation) to construct smooth (say in H^k, k large) initial data which are arbitrary small in $L^\infty(\mathbb{R}^d)$ and lead to solutions which have very large values at a specified, dispersive blow-up point.

Same result for the NLS in one dimension :

$$iu_t + u_{xx} \pm |u|^p u = 0$$

with $1 \leq p < 3$.

- Use Duhamel formula and Strichartz to control the integral term.

Take $(x^*, t^*) = (0, \frac{1}{4})$. Let $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$ be one of the initial data which leads to dispersive blow-up at the point $(x, t) = (0, \frac{1}{4})$. By Duhamel's formula

$$\begin{aligned} u(x,t) &= S(t)u_0 + C\int_0^t \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{1}{2}}} e^{i\frac{(x-y)^2}{4(t-s)}} |u|^p u(y,s)dyds && (5)\\ &=: S(t)u_0 + CI(x,t), && (6) \end{aligned}$$

where C is a non-zero constant. It is enough to check that $I(x,t)$ in (5) is a continuous function of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. This will follow from Lebesgue's theorem as soon as the integral is known to be locally bounded.

By Hölder's inequality, for all $x \in \mathbb{R}$,

$$
\begin{aligned}
|I(x,t| &\leq \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} ||u(.,s)||_{L_x^{p+1}}^{p+1} ds| \qquad (7)\\
&\leq \left(\int_0^t \frac{ds}{(t-s)^{\frac{\gamma}{2}}}\right)^{\frac{1}{\gamma}} \left(\int_0^t ||u(.,s)||_{L_x^{p+1}}^{\gamma'(p+1)} ds\right)^{\frac{1}{\gamma'}} \qquad (8)
\end{aligned}
$$

where $\gamma \in (1,2)$ and $\frac{1}{\gamma'} = 1 - \frac{1}{\gamma}$ will be fixed presently. The standard Strichartz estimates assert that for any $T > 0$, $u \in L^q(0,T; L^r(\mathbb{R}))$, where q and r are admissible. Take $r = p+1$ (implying $p \geq 1$). The corresponding q is $q = \frac{4(p+1)}{p-1}$. The condition $\gamma'(p+1) \leq q$ leads to $\gamma' \leq \frac{4}{p-1}$, which is to say, $\frac{1}{\gamma'} = 1 - \frac{1}{\gamma} \geq \frac{p-1}{4}$, or $\gamma \geq \frac{4}{5-p}$. This choice is compatible with $\gamma < 2$ if and only if $p < 3$. Thus, for $p < 3$, the integral I is a bounded function of x and t in $\mathbb{R} \times [0,T]$, for any $T > 0$.

The rogue waves phenomenon in optics is modeled by the following NLS type equation :

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{\alpha}{2} A \quad &- \quad \sum_{k \geq 2} \frac{i^{k+1}}{k!} \beta_k \frac{\partial^k A}{\partial z^k} \\ &= \quad i\gamma \left(1 + i\tau_{shock} \frac{\partial}{\partial z}\right) \left(A(z,t) \int_{-\infty}^{+\infty} R(z')|A(z',t)|^2 dz'\right). \end{aligned}$$

In this generalized Schrödinger equation, the dispersion is represented by its Taylor series and the nonlinearity features what is usually called a response function of the form $R(z) = (1 - f_R)\delta + f_R h_R(z)$, where δ is the Dirac mass. Thus the nonlinearity generally includes both instantaneous electronic and delayed Raman contributions.
Sketched here is a proof that dispersive blow-up also occurs in this model, thus providing a rigorous account of a possible explanation of the formation of optical rogue wave formation.

Consider first the linear part and for convenience, truncate the Taylor expansion of the dispersion so the linear model becomes

$$\begin{cases} \dfrac{\partial A}{\partial t} + \dfrac{\alpha}{2} A - \displaystyle\sum_{2 \le k \le K} i^{k+1} \gamma_k \frac{\partial^k A}{\partial z^k} = 0, \\ A(x,0) = A_0(x) \end{cases} \tag{9}$$

where $\gamma_K \neq 0$. By changing the independent variable from A to $B = e^{-\alpha t} A$, one may take it that the damping coefficient α is zero.

Demonstrating dispersive blow-up for the linear equation (9) can be reduced (by perturbation arguments very similar to those used below for the linearized water-wave equation) to showing dispersive blow-up for the linear equation with *homogeneous* dispersion, *viz.*

$$\begin{cases} \dfrac{\partial A}{\partial t} - i^{K+1}\dfrac{\partial^K A}{\partial z^K} = 0, \\ \\ A(x,0) = A_0(x), \end{cases} \tag{10}$$

where γ_K is set equal to 1 without loss of generality. Equation (10) specializes to the Airy and the linear Schrödinger equations as particular cases when $K = 3$ and 2, respectively. When $K \geq 4$ one can use Sidi-Sidi-Sulem or BenArzti-Koch-Saut to evaluate the corresponding fundamental solution and then construct suitable smooth initial data (a weighted version of the fundamental solution) which leads to dispersive blow-up.

This linear theory may then be extended to the nonlinear case. When the coefficient $\tau_{shock} = 0$, the equation is "semilinear" and the result follows by using Strichartz estimates as for the one-dimensional nonlinear Schrödinger equation. (This is especially transparent when the instantaneous electronic contribution vanishes, that is when $f_R = 1$, but it holds without this restriction.)

When $\tau_{shock} \neq 0$, the nonlinear term involves a derivative with respect to z. Assume now that $K \geq 3$. The crux of the matter is to analyze the double integral term in the Duhamel representation of the solution and to show that it defines a continuous function of space and time. When $K = 3$ we are reduced to the Korteweg de Vries case which was dealt with already in Bona-S. '93 by using a theory of the Cauchy problem in suitable weighted L^2-spaces. This analysis was also extended in Bona-S. '93 to a class of fifth order Korteweg–de Vries equations. This extension is easily made for any odd value of K greater than 7.

When $K \geq 4$ is even, the equation is of Schrödinger type and the weighted space theory (which uses in a crucial way that the phase velocity of the linear equation has a definite sign) does not appear to work. One has to rely instead on the higher-order smoothing properties of the linear group that appertain to the higher-order dispersion.

"Motivated" by the freak (rogue) wave question we consider here the linearized (at the trivial state) water wave system in one or two spatial dimension :

$$\begin{cases} \eta_{tt} + \omega^2(|D|)\eta = 0 \quad \textit{in} \quad \mathbb{R}^d \times \mathbb{R}_+, \\ \eta(.,0) = \eta_0, \quad \eta_t(.,0) = \eta_1. \end{cases} \tag{11}$$

Here $d = 1,2$ and $\omega^2(|D|) = g|D|\tanh(h_0|D|)$.
We consider the finite depth case and will scale the equations so that the gravity constant g and the mean depth h_0 are equal to 1.
The solution is explicit via Fourier transform ($\mathbf{k} = (k_1, k_2)$) :

$$\hat{\eta}(\mathbf{k},t) = \hat{\eta}_0(\mathbf{k})\cos[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}] + \frac{\sin[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}]}{(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}}\hat{\eta}_1(\mathbf{k}).$$

- Obviously well-posed in L^2.

Look for well/ill-posedeness in L^∞. Taking $\eta_1 \equiv 0$ we are reduced to proving that, for a fixed $t > 0$, $m_t(\mathbf{k}) = e^{it(\mathbf{k}\tanh(k))^{\frac{1}{2}}}$ is/(is not) a Fourier multiplier in L^∞. Recall that
$\{Fourier \quad multipliers \quad in \quad L^\infty\} =$
$\{Fourier \quad transforms \quad of \quad bounded \quad measures\}$.
Noticing that $(\mathbf{k}\tanh(\mathbf{k}))^{\frac{1}{2}} = |\mathbf{k}|^{\frac{1}{2}}(1 - \frac{2}{1+e^{2|\mathbf{k}|}})^{\frac{1}{2}} \equiv |\mathbf{k}|^{\frac{1}{2}} + r(\mathbf{k})$, we have

$$e^{it(\mathbf{k}\tanh(\mathbf{k}))^{\frac{1}{2}}} = (1 + f_t(\mathbf{k}))e^{it|\mathbf{k}|^{\frac{1}{2}}},$$

where

$$f_t(\mathbf{k}) = -2\sin\frac{tr(\mathbf{k})}{2}[\sin\frac{tr(\mathbf{k})}{2} - i\cos\frac{tr(\mathbf{k})}{2}]$$

is continuous, smooth on $\mathbb{R}^d \setminus \{0\}$, and decays exponentially to 0 as $|\mathbf{k}| \to \infty$, *uniformlyonboundedtimeintervals*

Theorem

Let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$, $d = 1, 2$ be given. There exists $\eta_0 \in C^\infty(\mathbb{R}^d \setminus \{0\}) \cap C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the solution $\eta \in C_b(\mathbb{R}; L^2(\mathbb{R}^d))$ of (LWW) with $\eta_1 \equiv 0$ is such that

(i) η is a continuous function of x and t on $\mathbb{R}^d \times \big((0, +\infty) \setminus \{t^*\}\big)$,

(ii) $\eta(\cdot, t^*)$ is continuous in x on $\mathbb{R}^d \setminus \{x*\}$,

(iii) $\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\(x,t)\neq(x^*,t^*)}} |\eta(x,t)| = +\infty.$

One may assume that $(x^*, t^*) = (0, 1)$. The idea is to take $\eta_1 = 0$ and $\eta_0(x) = |x|^\lambda \bar{K}(x)$, where $\frac{3d}{2} \leq \lambda \leq 2d$, and

$$K = \mathcal{F}^{-1}\left(\psi(|\mathbf{k}|)e^{i|\mathbf{k}|^{\frac{1}{2}}}\right),$$

where $\psi \in C^\infty(\mathbb{R}), 0 \leq \psi \leq 1, \psi \equiv 0$ on $[0, 1], \psi \equiv 1$ on $[2, +\infty)$.

We use a classical result (Wainger 1965, Miyachi 1981, and Hardy 1913 for $n = 1$) on the precise behavior of $\mathfrak{F}(\psi(|\mathbf{k}|)e^{i|\mathbf{k}|^a})(x)$ at 0 which we state in $\mathbb{R}^n$:

Theorem

(Wainger, Miyachi)
Let $0 < a < 1$, $b \in \mathbb{R}$ and define
$F^{\epsilon}_{a,b}(\mathbf{x}) =: \mathcal{F}\left(\psi(|\mathbf{k}|)|\mathbf{k}|^{-b}\exp(-\epsilon|\mathbf{k}| + i|\mathbf{k}|^{a})\right)(\mathbf{x})$ *for $\epsilon > 0$ and $\mathbf{x} \in \mathbb{R}^d$. The following is true of the function $F^{\epsilon}_{a,b}$.*
(i) $F^{\epsilon}_{a,b}(\mathbf{x})$ depends only on $|\mathbf{x}|$.
(ii) $F_{a,b}(\mathbf{x}) = \lim_{\epsilon\to 0^+} F^{\epsilon}_{a,b}(\mathbf{x})$ exists pointwise for $\mathbf{x} \neq 0$ and $F_{a,b}$ is smooth on $\mathbb{R}^d \setminus \{0\}$.
(iii) For all $N \in \mathbb{N}$, and $\mu \in \mathbb{N}^d$, $|\left(\frac{\partial}{\partial x}\right)^{\mu} F_{a,b}(\mathbf{x})| = \mathcal{O}(|\mathbf{x}|^{-N})$ as $|\mathbf{x}| \to +\infty$.
(iv) If $b > d(1-\frac{1}{2})$, $F_{a,b}$ is continuous on $\mathbb{R}^d$.
(v) If $b \leq d(1-\frac{1}{2})$, then for any $m_0 \in \mathbb{N}$, the function $F_{a,b}$ has the asymptotic expansion

$$F_{a,b}(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{\frac{1}{1-a}(d-b-\frac{ad}{2})}} \exp\left(\frac{i\xi_a}{|\mathbf{x}|^{\frac{a}{1-a}}}\right) \sum_{m=0}^{m_0} \alpha_m |\mathbf{x}|^{\frac{ma}{1-a}} + 0(|\mathbf{x}|)^{\frac{(m_0+1)a}{(1-a)}} + g(\mathbf{x}) \quad (12)$$

as $\mathbf{x} \to 0$, where $\xi_a \in \mathbb{R}$, $\xi_a \neq 0$, and g is a continuous function.

Use this result with $a = 1/2$ and $b = 0$. The choice of λ ensures that $(0, 1)$ is a blow-up point and that the other values of (x, t) are under control.

Remark

The phase velocity $g^{\frac{1}{2}}(\frac{\tanh(|\mathbf{k}|h_0)}{|\mathbf{k}|})^{\frac{1}{2}}\hat{\mathbf{k}}$ is a bounded function of $\mathbf{k}$. This is contrary to the case of the linear KdV-equations (Airy-equation) and the linear Schrödinger equation, where both the phase velocity and the group velocity become unbounded in the short wave limit. It is also unlike the case of linear gravity waves on the surface of an infinite layer of fluid, where the phase velocity is unbounded in the long wave limit. The dispersive blow-up phenomenon observed here is thus not linked to the unboundedness of the phase velocity.

Consider now the case of the linearized gravity-capillary waves. Again taking all the physical constants equal to 1 to simplify the discussion, the solution of (LWW) becomes

$$\begin{aligned}\hat{\eta}(\mathbf{k},t) &= \hat{\eta}_0(\mathbf{k})\cos\left[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}\right] \\ &+ \hat{\eta}_1(\mathbf{k})\frac{\sin\left[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}\right]}{(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}}. \quad (13)\end{aligned}$$

From this formula, it is readily discerned that for $(\eta_0,\eta_1)\in H^k(\mathbb{R}^d)\times H^{k-\frac{3}{2}}(\mathbb{R}^d)$, $k\in\mathbb{N}$, (LWW) has a unique solution $\eta\in C_b(\mathbb{R};H^k(\mathbb{R}^d))\cap L^2_{loc}(\mathbb{R};H^{k+\frac{1}{4}}_{loc}(\mathbb{R}^d))$.

Theorem

For $d = 1$ or 2, let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$, be given. There exists $\eta_0 \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the solution $\eta \in C_b(\mathbb{R}; L^2(\mathbb{R}^d))$ of (LWW) with $\eta_1 \equiv 0$ satisfies

(i) η is a continuous function of x and t on $\mathbb{R} \times \big((0, +\infty) \setminus \{t^*\}\big)$,

(ii) $\eta(\cdot, t^*)$ is continuous in x on $\mathbb{R} \setminus \{x*\}$, and

(iii)

$$\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\(x,t)\neq(x^*,t^*)}} |\eta(x, t)| = +\infty.$$

- Possible link with rogue waves.

In all the previous situations one can use the DBU construction to find open sets $\mathcal{U}$ in $H^k(\mathbb{R}^d), k \geq 3$, such that if initial data u_0 is taken from $\mathcal{U}$, then $|u_0|_\infty \leq \epsilon$, but the corresponding solution u with u_0 as initial data has the property that $|u(\cdot, t^*)|_\infty \geq M$, where the positive values of ϵ, M and t^* are specified.

Similar results for Fractional Schrödinger equations

$$\begin{cases} iu_t + (-\Delta)^{\frac{a}{2}} u = 0, \quad 0 < a < 1, \\ u(\cdot, t) = u_0(\cdot). \end{cases} \tag{14}$$

for $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$. These equations occur as the linearization of some weak turbulence models (Zakharov *et al*).

- This type of blow-up is very different (and probably physically more relevant), than the well-known *nonlinear* blow-up for the critical or supercritical KdV or Schrödinger equations.
- It would be interesting (and important to precise the link with freak waves) to extend those results to the full water waves system, and in a first step to the linearized system around a non flat surface.
- Link with some recent work of V. Banica and L. Vega ?

- **Self-similar blow-up in the infinite mass case** :

1. (J.M. Ghidaglia, S. Jaffard, unpublished). Let $u(x,t) = \frac{1}{t^{\frac{1}{6}}} Ai^2(\frac{x}{t^{\frac{1}{3}}})$.
   u is smooth on $D = \{(x,t), x \in \mathbb{R},\ t > 0\}$.
   Using the Airy equation
   $$Ai''(z) - zAi(z) = 0,$$
   one checks that u solves
   $$\partial_t u + \partial_{xxx} u = 0, \quad \text{in} \quad D.$$
   Furthermore, for any fixed $t > 0$ $u(\cdot,t) \in L^p(\mathbb{R})$ for any $p > 2$ and
   $$||u(\cdot,t)||_p^p = \frac{C}{t^{\frac{p}{6}-\frac{1}{3}}} \int_{\mathbb{R}} Ai^{2p}(x)dx.$$

Thus, for any $p > 2$ the L^p norm of $u(\cdot, t)$ blows up at $t = 0$ like $\frac{C}{t^{\frac{1}{6}-\frac{3}{p}}}$.
In particular the sup-norm blows up like $\frac{C}{t^{\frac{1}{6}}}$.

1. J. Bona, F. Weissler : Self-similar blow-up of complex-valued solutions to KdV (2001, 2009).
2. Gadi Fibich (Physica D 2011) :

$$i\psi_t + \Delta\psi + |\psi|^{2\sigma}\psi = 0, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+,$$

where $1 < \sigma d < 2$. Let $p^* = \frac{\sigma d}{\sigma d - 1}$. Construction of a solution (by self-similarity from the ground state) with
$\lim_{t \to T_c} ||u(\cdot, t)||_p = +\infty$, for any $p^* < p \leq \infty$.
This solution is not in L^2.